

# Phonon contribution to the shear viscosity of a superfluid Fermi gas in the unitarity limit

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We present a detailed analysis of the contribution of small-angle Nambu-Goldstone boson (phonon) collisions to the shear viscosity,  $\eta$ , in a superfluid atomic Fermi gas close to the unitarity limit. For low temperatures we show that the experimental values of the shear viscosity coefficient to entropy ratio,  $\eta/s$ , can be reproduced assuming that  $\eta$  is dominated by phonon collisions and considering an effective shear viscosity which takes into account the finite size of the experimental system. In particular, for very low temperatures,  $T \lesssim 0.1T_F$ , we find that phonons are ballistic and the shear viscosity is determined by the processes that take place at the interface between the superfluid and the normal phase. This result is independent of the detailed form of the phonon dispersion law and leads to two testable predictions: the shear viscosity should correlate with the size of the optical trap and it should decrease with decreasing temperature. For higher temperatures the detailed form of the phonon dispersion law becomes relevant and, within our model, we find that the experimental data for  $\eta/s$  can be reproduced assuming that phonons have an anomalous dispersion law.

## I. INTRODUCTION

The study of the transport coefficients of a fluid opens a window on its microscopic dynamics, helping us to understand which are the underlying degrees of freedom and the most important interaction channels [1–3]. Of extraordinary interest is the study of the transport properties of fluids with an infinite two-body scattering length (corresponding to the so-called unitarity limit), because the expression of the transport coefficients in terms of the thermodynamic variables is believed to be universal [4], meaning that it is independent of the detailed form of the inter-particle potential.

Experiments with trapped cold atomic gases are able to reach the region of infinite scattering length. When these experiments are performed with fermionic atoms, like  $^6\text{Li}$  or  $^{40}\text{K}$ , prepared in two different hyperfine states having a magnetic-field Feshbach resonance, the strength of the interaction between atoms with opposite spins can be varied by tuning the intensity of the magnetic field. At very low temperature, by varying the magnetic-field controlled interaction, fermionic pairing is observed to undergo the Bose-Einstein condensate (BEC) to Bardeen-Cooper-Schrieffer (BCS) transition, allowing us to explore the crossover region in a controlled way [5–8]. Given the adjustability of the inter-particle interaction, ultracold fermionic systems can serve to emulate non-relativistic systems (which have the same spontaneously broken symmetries of the ultracold fermionic system) and to determine the universal expression of the transport coefficients at unitarity. On a more general ground, the results obtained can be used to emulate the properties of different systems which are believed to have similar properties as those of the Fermi gas at unitarity. One remarkable example is the fluid realized in relativistic heavy-ion collisions, where almost ideal hydrodynamic flow has been measured [2].

Cold atomic systems at unitarity can also be used to explore the conjecture derived by string theory methods [9], of the existence of a universal bound of the shear viscosity to entropy ratio  $\eta/s \geq \hbar/(4\pi k_B)$ . The fluid produced in heavy-ion collisions seems to have extremely small shear viscosity [1], close to the universal bound, while it is not yet clear which is the lower bound of  $\eta/s$  in fermionic cold atoms. As we shall see in detail, the present experimental results give values of  $\eta/s$  above the universal bound, but the data seem to indicate that this quantity would further decrease with decreasing temperature. Recently, other different lower bounds on  $\eta/s$  have been predicted based on the study of the effect of hydrodynamical fluctuations [10, 11].

In the present paper we focus our attention to the superfluid phase of the ultracold Fermi liquid at unitarity. In superfluid Fermi systems the low energy degrees of freedom are the Nambu-Goldstone bosons (NGBs), which arise from

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the quantum condensation of difermions by a process that spontaneously breaks a global continuous symmetry of the system. The property of superfluidity follows from the fact that the NGBs at low energy have a linear dispersion law. We will refer generically to these modes as superfluid phonons, or phonons for simplicity. At very low temperatures they are the only relevant degrees of freedom, as the fermions are gapped and thermally suppressed. Thus, phonons give the leading thermal contribution to the thermodynamic and hydrodynamic properties of the ultracold superfluid, and the transport properties of the system strongly depend on the phonon self-interactions.

Since the transport properties of the ultracold superfluids are very sensitive to the phonon Lagrangian, it is possible to have hints on the phonon dispersion law and on the most important interaction channel by appropriate experimental measurements. In one of his pioneering works, Landau proposed a particular phonon spectrum [12], in order to explain the experimental values of the thermodynamic functions of  $^4\text{He}$ . An analogous low energy spectrum can be inferred from the study of the transverse phonon relaxation processes, see e.g. [13]. Also the most relevant phonon self-interactions valid for  $^4\text{He}$  can be determined in a similar way [14, 15].

More recently it has been realized that the effective field theory (EFT) techniques can be used to determine the phonon dispersion law as well as the main phonon self-interactions for many superfluids [16]. The EFTs are very efficient when there is a hierarchy of widely separated energy or momentum scales (in our system, the phonon momentum versus the Fermi momentum), which allows one to obtain an expansion of the effective phonon Lagrangian in powers of energy or momentum, rather than in a coupling constant. Power counting and symmetry considerations fix the form of the EFT to the accuracy one desires, regardless of whether the underlying system is weakly or strongly coupled. At leading order in a momentum expansion, one can see that the EFT phonon Lagrangian is related to Landau's Hamiltonian by a Legendre transformation [16]. Using the power of these EFT one can thus relate the physics of phonons in different superfluid systems having the same global symmetries.

In this manuscript we give an extended explanation of the results of Ref. [17], where we employed both the EFT techniques as well as the experimental available data, to pin the phonon Lagrangian down. In our procedure we assume a generic expression of the phonon effective Lagrangian, obtained by an expansion in powers of the phonon momentum, and study the contributions of various phonon processes to the shear viscosity coefficient of a superfluid ultracold Fermi liquid at unitarity. For the evaluation of  $\eta$ , we consider the Beliaev  $1 \leftrightarrow 2$  processes, using for this purpose the scattering rates obtained from the EFT Lagrangian of Ref. [16]. We perform this calculation assuming an infinite volume and employing the results for the phonon spectrum obtained by various techniques [18, 19]. Our results, and associated discussion, can be easily related to those obtained for  $^4\text{He}$  in the temperature,  $T$ , regime where the viscosity is dominated by the phonons [14, 15]. The resulting expression of the shear viscosity coefficient depends on one free parameter, which is related to the phonon dispersion law at high momentum and that has not been yet evaluated. Then, we try to fix this free parameter by comparing our results with those obtained by the Duke group [20, 22], where measurements of the entropy of ultracold fermionic systems and of the shear viscosity coefficient (by measuring the damping of the breathing mode) have been obtained. Unfortunately reaching temperatures much smaller than the Fermi temperature,  $T_F$ , is extremely difficult and only few experimental points are available at the temperature where the contribution of phonons might be relevant. Moreover, the experimental setup is such that finite size effects must be included for an appropriate comparison with our results. Indeed, for  $T \sim 0.1 T_F$ , we find that for the particular experimental setup of Refs. [20, 22] phonons are ballistic, because their mean free path is of the same order of the size of the trapped cloud. Thus, the infinite volume limit we used in the evaluation of the shear viscosity coefficient and the whole hydrodynamical treatment of the phonons ceases to be valid. However, in the ballistic regime the collisions of phonons with the boundary still produce dissipation, and ultimately, the damping of the breathing mode. This dissipation can be described by an effective ballistic shear viscosity, employing an approach very similar to the one used to describe the dissipation in the ballistic regime of the phonons of  $^4\text{He}$ , or other systems [23–25].

Once the parameters of the effective Lagrangian of phonons are known, they can be used to predict the behavior of different transport properties at very low temperature, such as the thermal conductivity [26], or the bulk viscosity coefficients [27]. In the present paper we content ourselves to consider the behavior of  $\eta/s$  at very low temperature, marginally reached by today experiments, and to predict the behavior of this quantity with decreasing temperature. Indeed, according to our calculations, trapped phonons below  $\sim 0.1 T_F$  are ballistic, and it follows that  $\eta/s$  should be directly proportional to the temperature. We also find that  $\eta$  should approximately scale with the size of the trap, a behavior that future experiments should be able to check as well.

This paper is structured as follows. In Sec. II we present a brief review of the phonon EFT at leading and next-to-leading order in the momentum expansion, with emphasis on the expression of the phonon dispersion law. We use the EFT to get the scattering rates necessary for the evaluation of the contribution to  $\eta$  of small-angle collisions in Sec. II A. In Sec. III A we use a variational approach to solve the Boltzmann equation for three phonon collisions, and obtain their contribution to  $\eta$ . The  $T$  dependence of such a contribution strongly depends on the form of the phonon dispersion law, and we obtain different values of  $\eta$  for some specific choices of the dispersion law in Sec. III B. In Sec. IV we discuss the experimental settings of the trapped Fermi gases where the measurements of  $\eta$  have been done.

In Sec. IV A we compute the phonon mean free path and compare it with the size of the atomic cloud, to reach to the conclusion that finite size effects have to be taken into account. We write a phenomenological formula for  $\eta$  that takes into account those effects, and use it in Sec. IV B to give account of the experimental measures of  $\eta/s$ . Appendix A discusses why it is enough to keep the lower order corrections in the phonon dispersion law in the evaluation of  $\eta$ , while in Appendix B we present an alternative variational solution to the Boltzmann equation. Throughout the paper we use natural units  $\hbar = k_B = 1$ .

## II. PHONON COLLISIONS RELEVANT FOR THE VISCOSITY IN THE FERMI SYSTEM AT UNITARITY

We call the superfluid phonon the NGB associated to the spontaneous symmetry breaking of a global  $U(1)$  symmetry group associated to particle number conservation. The phonon field can be viewed, by a microscopic approach, as the phase oscillation of the difermion condensate when the fermionic degrees of freedom have been integrated out. Actually, in this perspective, the low energy spectrum contains also the degree of freedom associated to the oscillation of the absolute value of the difermion condensate, see e.g. [28–30]. But considering only gaussian fluctuation, the radial degree of freedom can be integrated out and only results in a modification of the phonon dispersion law. In the following we assume that in the EFT Lagrangian the radial oscillations of the condensate have already been integrated out and the phonon is the only dynamical degree of freedom. It is worth to note that the fluctuations in the absolute value of the condensate are not taken into account in the analysis of [31, 32] where the Lagrangian of the collective modes has been determined starting from a mean-field Hamiltonian.

Effective field theory techniques can be used to derive the Lagrangian of the superfluid system at unitarity by a derivative expansion in the phonon field. At the lowest order, the effective Lagrangian has been obtained in Ref. [16] and is formally given at vanishing temperature by the functional

$$\mathcal{L}_{\text{LO}} = P(X) , \quad (1)$$

with

$$X = \mu_0 - V(\mathbf{r}) - \partial_t \varphi - \frac{(\nabla \varphi)^2}{2m} , \quad (2)$$

where  $P(\mu_0)$  and  $\mu_0$  are the pressure and chemical potential, respectively, of the non relativistic superfluid at  $T = 0$ ,  $\varphi$  is the phonon field, and  $V$  is the trapping potential. For  $V = 0$ , the equation of state (EoS) of the unitary Fermi gas reads

$$P = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}} m^{3/2} \mu_0^{5/2} , \quad (3)$$

where  $m$  is the mass of the particles that condense and  $\xi \simeq 0.36 - 0.38$  [33–35] is the Bertsch number [36], a universal constant that fixes the relation between chemical potential and Fermi energy,  $\mu_0 = \xi E_F$ .

It is convenient to re-express the LO Lagrangian in a different way. After a derivative expansion, and rescaling of the phonon field to have a canonically normalized kinetic term, one obtains

$$\mathcal{L}_{\text{LO}} = \frac{1}{2} ((\partial_t \phi)^2 - v_{\text{ph}}^2 (\nabla \phi)^2) - g ((\partial_t \phi)^3 - 3g_s \partial_t \phi (\nabla \phi)^2) + \mathcal{O}((\partial \phi)^4) + \dots . \quad (4)$$

where we have set  $V = 0$  for simplicity. The different self-coupling constants of Eq. (4) can be expressed in terms of derivatives of the pressure with respect to the chemical potential [37]. For comparison with the condensed matter literature on superfluidity, and for the purposes of computation, it turns out to be more convenient to express them in terms of the density  $\rho$ , the speed of sound at  $T = 0$ ,  $c_s$ , and derivatives of the speed of sound with respect to the particle density (see Appendix A of Ref. [37])

$$v_{\text{ph}} = c_s = \sqrt{\frac{1}{m} \frac{\partial P}{\partial \rho}} , \quad g = \frac{1}{6\sqrt{m\rho} c_s} (1 - 2u) , \quad g_s = \frac{c_s^2}{1 - 2u} , \quad (5)$$

where  $u = \frac{\rho}{c_s} \frac{\partial c_s}{\partial \rho}$  is the so-called Grüneisen number.

The next-to-leading order (NLO) Lagrangian is constructed by demanding non-relativistic general coordinate invariance and conformal symmetry [16], and it reads

$$\mathcal{L}_{\text{NLO}} = c_1 m^{1/2} \frac{(\nabla X)^2}{\sqrt{X}} + \frac{c_2}{\sqrt{m}} (\nabla^2 \varphi)^2 \sqrt{X} , \quad (6)$$

where  $c_1$  and  $c_2$  are two dimensionless parameters, which are universal but cannot be determined by symmetry considerations. Both coefficients determine not only the corrections to the different phonon self-couplings in Eqs.(5), but, and this is more important for the evaluation of the shear viscosity, also the corrections to the phonon dispersion law which at the NLO reads [16]

$$E_k = c_s k(1 + \gamma k^2), \quad \gamma = - \left( c_1 + \frac{3}{2} c_2 \right) \frac{\pi^2 \sqrt{2\xi}}{k_F^2}, \quad (7)$$

where  $k_F = \sqrt{2mE_F}$  is the Fermi momentum. The sign of  $\gamma$  has a dramatic effect on the possible phonon interaction channels, because, as we shall discuss in detail in the next section, the Beliaev process  $1 \leftrightarrow 2$  between massless particles is only allowed for positive values of  $\gamma$ . The numerical values of  $c_1$  and  $c_2$  are still uncertain, but some progress has been made by a number of different techniques. Both coefficients can be estimated by the  $\epsilon$ -expansion technique, which gives  $c_1 + 3/2c_2 \simeq -0.0209$  [18]. However, these estimates include large errors, in particular for the determination of  $c_2$ , because a finite value of  $c_2$  only appears at the second order in the  $\epsilon$  expansion [18]. A different approach is presented in [19] where the phonon dispersion law is determined by a fit of Quantum Monte-Carlo (QMC) simulations. The reported result is  $c_1 + 3/2c_2 \simeq -0.028$ , but large uncertainties are present, arising from the numerical fit of the QMC data. A mean-field theory has been employed in Ref. [38], giving  $c_1 + 3/2c_2 \simeq -0.0205$ . Although with large numerical uncertainties all these methods give results consistent with a positive  $\gamma$  coefficient. A different estimate of these parameters can be obtained by means of a mixed approach [39], which combines the  $\epsilon$ -expansion technique and the results obtained considering the gaussian fluctuations on the top of the mean field solution [30]. According to [39],  $c_2$  is proportional to the square of the pair size and should therefore be positive (in disagreement with the estimate of [38]) and presumably comparable in size to  $c_1$ . The results of [39] lead to  $c_1 + 3/2c_2 \simeq +0.01$  and therefore to a negative value of  $\gamma$ , in disagreement with [18, 19, 38]. However, considering gaussian fluctuations close to the unitarity might not be enough to determine the dispersion law of the collective modes, because the procedure seems to be not self-consistent [30]. For this reason we assume that

$$\gamma \simeq \frac{0.18}{k_F^2}, \quad (8)$$

which is obtained considering  $c_1 + 3/2c_2 = -0.021$  and  $\xi = 0.37$ . We shall then briefly comment on how our results would change for negative values of  $\gamma$ .

Given the expression of  $X$  in Eq.(2), it is clear that in the NLO Lagrangian, not only the trapping potential appears, but also its space derivatives. In order to simplify the calculations we will use the local density approximation (LDA), which corresponds to consider the system as locally homogeneous, and therefore we will neglect the spatial derivatives of the trapping potential. Approximating the trapping potential by the harmonic potential

$$V(\mathbf{x}) = \frac{1}{2} m \omega_i^2 x_i^2, \quad (9)$$

the LDA turns out to be a good approximation as far as the local value of the chemical potential is much larger than  $\omega_i$ , which is a condition that is satisfied in the experimental settings of Ref. [20, 22], except near the edge of the cloud. In the following we will first consider the situation where there is no trapping potential,  $V = 0$ , and then comment on the effects in the phonon physics due to the presence of the trap.

Terms in the phonon Lagrangian beyond the NLO will as well change the phonon dispersion law and the interaction vertices. We are not aware, to the best of our knowledge, of a determination of the phonon dispersion law at higher orders<sup>1</sup>. In general, the phonon dispersion law can be written as

$$E_k = c_s k(1 + \psi(k)), \quad (10)$$

where  $|\psi(k)| \ll 1$  and it can be Taylor expanded in  $k^2/k_F^2$ ,

$$\psi(k) = \gamma k^2 + \delta k^4 + \mathcal{O}\left(\frac{k^6}{k_F^6}\right), \quad (11)$$

where  $\delta \propto 1/k_F^4$  and we shall neglect terms of order higher or equal to  $k^6/k_F^6$ . We shall obtain a numerical estimate of  $\delta$  from a fit of the experimental values of the shear viscosity to entropy ratio. In doing this we shall follow a

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<sup>1</sup> See, however, the results of Ref. [31], which refer to the dispersion law of hydrodynamical sound waves.

procedure akin to the one used in  $^4\text{He}$ , where the spectrum of the low energy excitations is determined from a fit of the experimental values of the mean free path of phonons, see e.g. [13]. We comment on the appropriate temperature range of validity of the expansion in Eq.(11) in the Appendix A, where also an estimate of the  $k^6/k_F^6$  term is given.

In principle, from a naive power counting, one would expect that NLO (or higher order) corrections to the interaction vertices should be taken into account as well. However, we shall show that in the computation of the shear viscosity coefficient these corrections are subleading with respect to the one determined by the dispersion law. The reason being that, as we shall discuss in detail in the next section, the dispersion law determines the power dependence of the shear viscosity coefficient on the temperature. On the other hand, the neglected terms in the interaction vertices do always give corrections of higher power in  $T/T_F$ .

### A. Three phonon and four phonon interactions

For the computation of the shear viscosity coefficient one needs to evaluate the scattering rates of both the binary (hereafter 4-ph) collisions and the Beliaev  $1 \leftrightarrow 2$  (hereafter 3-ph) processes. The vertices associated to the corresponding interactions can be obtained by the momentum expansion of the Lagrangian in Eq.(1), and then the scattering amplitudes are readily obtained by standard techniques.

Both the contributions of 3-ph and 4-ph processes to the shear viscosity depend on the dispersion law of the phonons. However, the 3-ph processes are more sensitive to the expression of  $\psi(k)$  because of kinematical restrictions. Considering the LO phonon dispersion law,  $E_p = c_s p$ , one finds that, for 3-ph processes, energy and momentum conservations only allow the collinear scattering, and the contribution of this process to the shear viscosity diverges. The reason being that the shear viscosity coefficient is proportional to the time scale for momentum transfer in the orthogonal direction to the flow, and if the collision is collinear it takes an infinite amount of time to transfer momentum in the orthogonal direction.

Considering the generic corrections to the LO Lagrangian, the dispersion law has an additional term  $\psi(k)$ , see Eq. (10), which may favor or disfavor  $1 \leftrightarrow 2$  processes depending on the sign of  $\psi(k)$ . If  $\psi(k) < 0$ , 3-ph processes are kinematically forbidden. On the other hand, if  $\psi(k) > 0$  (at least for some momenta  $k$ ), 3-ph processes are kinematically allowed. In the following we shall assume that the NLO corrections give  $\gamma > 0$  and therefore make  $\psi(k)$  positive for certain values of  $k$ .

If the three phonons participating in the collision have momenta  $\mathbf{p}$ ,  $\mathbf{p}'$  and  $\mathbf{k}$ , then the collision angle between the phonons with momenta  $\mathbf{p}$  and  $\mathbf{p}'$  is given by

$$\cos \theta_{pp'} = 1 + \frac{1}{p'} (p' \psi(p') + k' \psi(k') - (p' + k') \psi(p' + k')) , \quad (12)$$

and since  $\psi(k) \ll 1$ , one can deduce that the deflection angle is small, and roughly proportional to  $\theta \sim \psi^{1/2}$ . After an analysis of the scattering processes one obtains that phonons with thermal momentum close to  $k_{\text{th}} \sim 10T/c_s$  give the leading contribution to the viscosity, and one can estimate  $\theta_{\text{th}} \sim \psi_{\text{th}}^{1/2}$ , where  $\psi_{\text{th}} = \psi(k_{\text{th}})$ .

In the following analysis of the contribution of the 3-ph processes to the shear viscosity, we will be using a double expansion, both in the phonon momentum and in the collision angle. We will be considering the vertices of the scattering rates as arising from the lowest momentum expansion  $\mathcal{L}_{\text{LO}}$ , as corrections from  $\mathcal{L}_{\text{NLO}}$  would be suppressed in the shear viscosity by powers of  $T/(c_s k_F) \propto T/T_F$ . However, we will use the phonon dispersion law beyond LO, as the expression of  $\psi(k)$  determines the collision angle, and approximate all different expressions in small angles.

From the LO Lagrangian we have that the square of the scattering amplitude of the 3-ph processes is given by

$$|\mathcal{M}_{3\text{ph}}|^2 = 36g^2 (-E_p E_{p'} E_{k'} + g_s (E_p \mathbf{p}' \cdot \mathbf{k}' + E_{p'} \mathbf{p} \cdot \mathbf{k}' + E_{k'} \mathbf{p} \cdot \mathbf{p}'))^2 , \quad (13)$$

where  $g_s$  is defined in Eq. (5),  $E_p$ ,  $E_{p'}$  and  $E_{k'}$  are the energies of the phonons with momenta  $\mathbf{p}$ ,  $\mathbf{p}'$  and  $\mathbf{k}'$ , respectively. After using the constraints of energy and momentum conservation and considering phonons with the dispersion law in Eq. (10), the LO scattering amplitude can be expressed as

$$|\mathcal{M}_{3\text{ph}}|^2 = \frac{4c_s^4 (1+u)^2}{\rho} (pp'k')^2 + \mathcal{O}(\psi) . \quad (14)$$

This expression is formally equivalent to the scattering amplitude for the 3-ph processes in superfluid  $^4\text{He}$ , and it has been used by Maris [14] and Benin [15] for the computation of the shear viscosity in that system. What changes are the explicit expressions of  $\rho$  and  $c_s$  as well as the  $\mathcal{O}(\psi)$  corrections. The fact that the expressions of the 3-ph processes are the same in these two superfluid systems is not unexpected, as it was shown [16] that the formal expression of



the effective Lagrangian at LO, Eq. (1), can be related by a Legendre transformation to the Hamiltonian proposed by Landau to derive the phonon self-interactions in superfluid  $^4\text{He}$ .

The expression of the scattering amplitude of the 4-ph process has been derived at the LO in momentum expansion in Ref. [40]. We refer the reader to that paper for the corresponding expression of  $|\mathcal{M}_{4\text{ph}}|^2$ .

The results obtained above still hold when including the trapping potential in the local density approximation. The phonon effective field theory, and the associated scattering amplitudes of the 3-ph and 4-ph processes, are modified by taking into account that the  $x$ -dependence of the trapping potential can be included in a space dependent effective chemical potential,  $\mu(\mathbf{x}) = \mu_0 - V(\mathbf{x})$ . But this is not the only effect due to the presence of the trap. As we shall discuss in detail in Sec. IV, the restricted geometry dictated by the trapping potential leads to important boundary effects when the mean free path of phonons becomes comparable with the typical length scale of the trap.

### III. SHEAR VISCOSITY DUE TO SMALL-ANGLE COLLISIONS

The shear viscosity  $\eta$  measures the transport of momentum in the orthogonal direction to the hydrodynamic flow. It is generally dominated by large-angle collisions of the quasiparticle components of the fluid. There are cases, however, when small-angle collisions may dominate, as it may turn out to be more efficient to achieve a large-angle collision by the addition of many small-angle scatterings. In the low  $T$  regime of superfluid  $^4\text{He}$  where the viscosity is dominated by phonons, the 4-ph large-angle collisions give the leading contribution to the shear viscosity only in a restricted range of temperature [41], while 3-ph small-angle processes dominate for any temperature below 0.7 K [14].

In this section we use a kinetic theory approach to compute the 3-ph contribution to the shear viscosity, assuming that the system is infinite and homogeneous. We use the scattering rate as arising from  $\mathcal{L}_{\text{LO}}$  evaluated in the previous section, but assume in the energy and momentum conservation the generic phonon dispersion in Eq.(10), with  $\psi(k) > 0$ . We shall discuss in detail the range of validity of this method, that is why in the scattering rate the  $\mathcal{O}(\psi(k))$  corrections can be neglected. Eventually, we perform both an expansion in the phonon momentum and in the collision angle given in Eq. (12), finding that the temperature dependence of the shear viscosity strongly depends on the expression of the phonon dispersion law.

#### A. Variational solutions of the Boltzmann equation

When a shear stress is applied to a fluid, e.g. by contact with a sliding surface, it will perturb the fluid producing a gradient of the velocity field. In a superfluid, a sliding surface moving at sufficiently small velocity will only produce a gradient of the normal field velocity in the direction perpendicular to the surface. For small deviations from equilibrium, the energy-momentum tensor is given by

$$\delta T_{ij} = -\eta V_{ij} \equiv -\eta \left( \partial_i V_j + \partial_j V_i - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{V} \right), \quad (15)$$

where  $\mathbf{V}$  is the velocity of the normal fluid component.

At the microscopic level, the phonon contribution to the energy-momentum tensor of the system can be computed by means of kinetic theory and it is expressed by

$$T_{ij} = c_s^2 \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E_p} f(p, x), \quad (16)$$

where  $f(p, x)$  is the phonon distribution function, which obeys the Boltzmann equation [12]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial E_p}{\partial \mathbf{p}} \cdot \nabla f = C[f], \quad (17)$$

where we have assumed to be in the superfluid rest frame, and  $C[f]$  is the collision term. The processes that give the largest contribution to the collision term are 4-ph collisions and 3-ph splitting and joining processes. We shall assume that these processes can be evaluated separately (i.e. independently) and then compare the corresponding equilibration times. The contribution of the binary collisions to the shear viscosity has already been evaluated in Refs. [40, 42]. Therefore, here we shall focus on the 3-ph processes. For the  $1 \leftrightarrow 2$  processes the collision integral is

given by

$$C_{1\leftrightarrow 2}[f] = -\frac{1}{4E_p} \int_{p',k'} |\mathcal{M}_{3\text{ph}}|^2 (2\pi)^4 \delta^{(4)}(P - P' - K') \{f(p)(1+f(p'))(1+f(k')) - f(p')f(k')(1+f(p))\} \\ + \frac{1}{2E_p} \int_{p',k'} |\mathcal{M}_{3\text{ph}}|^2 (2\pi)^4 \delta^{(4)}(P' - P - K') \{f(p')(1+f(p))(1+f(k')) - f(p)f(k')(1+f(p'))\} , \quad (18)$$

where we have defined the shorthand notation

$$\int_p \equiv \int \frac{d^3p}{(2\pi)^3 2E_p} , \quad (19)$$

and the square of scattering amplitude  $\mathcal{M}_{3\text{ph}}$  was written in Eq. (13). As expected, the collision term vanishes when evaluated using the phonon Bose-Einstein equilibrium distribution function, so that  $C[f_{\text{eq}}] = 0$ .

For the computation of the transport coefficients we consider small departures from equilibrium so that the distribution function can be written as  $f = f_{\text{eq}} + \delta f$ , and then we linearize the transport equation in  $\delta f$ . For the computations of the shear viscosity, one assumes that

$$\delta f = -h(p)p_{kl}V_{kl} \frac{f_{\text{eq}}(1+f_{\text{eq}})}{T} , \quad (20)$$

where  $h(p)$  is an unknown function,  $V_{kl}$  is the tensor defined in Eq. (15), and we have defined

$$p_{kl} = p_k p_l - \frac{1}{3} \delta_{kl} p^2 . \quad (21)$$

Upon substituting the expression of the perturbed distribution function in Eq. (16) one can extract the value of the shear viscosity coefficient, which turns out to be

$$\eta_{3\text{ph}} = \frac{4c_s^2}{15T} \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{2E_p} f_{\text{eq}}(1+f_{\text{eq}})h(p) . \quad (22)$$

On the other hand, inserting Eq.(20) in the collision integral, we obtain at the linear order

$$\delta C \equiv \frac{1}{2E_p T} F_{ij}[h(p)]V_{ij} , \quad (23)$$

and the linearized Boltzmann equation reads

$$c_s \frac{f_{\text{eq}}(1+f_{\text{eq}})}{2pT} p_{ij}V_{ij} = \frac{1}{2E_p T} F_{ij}[h(p)]V_{ij} . \quad (24)$$

With this last result one realizes that the shear viscosity can also be written as

$$\eta_{3\text{ph}} = \frac{2}{5T} \int_p p_{ij}h(p)F_{ij}[h(p)] , \quad (25)$$

or equivalently,

$$\eta_{3\text{ph}} = \frac{1}{5T} \int_{p,p',k'} (2\pi)^4 \delta^{(4)}(P - P' - K') |\mathcal{M}_{3\text{ph}}|^2 f_p^{\text{eq}}(1+f_{p'}^{\text{eq}})(1+f_{k'}^{\text{eq}}) (\Delta_{ij}^h)^2 , \quad (26)$$

where

$$\Delta_{ij}^h \equiv h(p)p_{ij} - h(p')p'_{ij} - h(k')k'_{ij} , \quad (27)$$

and we have used the property that the scattering amplitude is symmetric in the momenta of the three phonons.

In order to have a lower bound on the shear viscosity coefficient it is possible to employ a variational method. In order to do so, one defines the inner product

$$\langle \phi_1 | \phi_2 \rangle = - \int_p \phi_1(p) \phi_2(p) \frac{df_{\text{eq}}}{dE_p} , \quad (28)$$

where  $|\phi_1\rangle$  and  $|\phi_2\rangle$  are two arbitrary functions. If we define the function  $|\chi\rangle = p_{ij}$ , then Eq.(24) can be schematically written as  $|\chi\rangle = H|\Phi\rangle$ , where  $\Phi$  is the solution of the linearized transport equation. The shear viscosity can be written as  $\eta_{3\text{ph}} = \langle\chi|\Phi\rangle$ , which is equivalent to Eq.(22), or alternatively,  $\eta_{3\text{ph}} = \langle\Phi|H|\Phi\rangle$ , as in Eq.(26). Finally, from the Schwarz inequality (and considering that  $H$  is positive semidefinite) it follows that for an arbitrary function  $\tilde{\Phi}$ ,

$$\eta_{3\text{ph}} \geq \frac{\langle\chi|\tilde{\Phi}\rangle^2}{\langle\tilde{\Phi}|H|\tilde{\Phi}\rangle}, \quad (29)$$

which is saturated when  $\tilde{\Phi}$  is the exact solution of the linearized transport equation.

We will compute the value of  $\eta_{3\text{ph}}$  using this variational method, and assuming that the phonon dispersion law is of the form given in Eq. (10), where  $\psi(k)$  is treated as a small perturbation. In principle an upper bound of the shear viscosity can as well be obtained, e.g. by the method proposed in [43]. However, from the fact that  $\eta_{3\text{ph}}$  should diverge for  $\psi(k) = 0$ , it is possible to figure out the particular family of functions which maximizes Eq.(29). More in detail, we look for a solution that minimizes  $(\Delta_{ij}^h)^2$ , and thus, considering the expression of  $H$ , maximizes the value of  $\eta_{3\text{ph}}$ . It is then easy to see that within the space of rational functions, the solution must be of the form

$$h(p) = \frac{1}{p} (1 + a\psi(p)) + \mathcal{O}(\psi^2), \quad (30)$$

where  $a$  is the variational parameter. This form of the solution is such that  $(\Delta_{ij}^h)^2 \propto \psi^2$ , while other forms of rational functions give a result of order one, and thus, bigger in the  $\psi$ -expansion. We fix the value of  $a$  by minimizing

$$\Delta_{ij}^h \Delta_{ij}^h = \frac{1}{9} (42 + 24a + 6a^2) (p'\psi(p') + k'\psi(k') - (p' + k')\psi(p' + k'))^2, \quad (31)$$

which occurs when  $a = -2$ . We have checked the stability of this variational solution by changing the form of variational functions used in the maximization procedure, see Appendix B. We have also checked that our variational solution agrees with the variational solution of the phonon transport equation given in Ref. [15], where the shear viscosity due to joining and splitting processes for superfluid  $^4\text{He}$  has been computed.

With the form of  $h(p)$  reported above, we computed the shear viscosity coefficient at the leading order in  $\psi$ , and we obtained

$$\eta_{3\text{ph}} = \left(\frac{2\pi}{15}\right)^4 \frac{T^8}{c_s^8} \frac{1}{M} + \mathcal{O}(\psi^2), \quad (32)$$

where

$$M = \frac{(1+u)^2}{20T\pi^3\rho} \int^* dp' dk' (p'k'(p'+k'))^2 (p'\psi(p') + k'\psi(k') - (p'+k')\psi(p'+k'))^2 f_{p'+k'}^{\text{eq}} (1 + f_{p'}^{\text{eq}})(1 + f_{k'}^{\text{eq}}), \quad (33)$$

and the star on top of the integral indicates that the integration is constrained by the energy and momentum conservation. As already remarked, the expression of the shear viscosity given above strongly depends on the form of the phonon dispersion law. We will explicitly check this fact in the following subsection by computing  $\eta_{3\text{ph}}$  for different choices of  $\psi(p)$ . Notice that in Eq.(33) we did not need to include in the scattering amplitude the  $\mathcal{O}(\psi)$  corrections. Actually, in Eq. (14) we did not even calculate such corrections. We shall see at the end of the next section that neglecting these correction is equivalent to neglect temperature corrections of order  $\mathcal{O}(T/T_F)^2$  in our final results.

## B. Values of the viscosity with different phonon dispersion laws

Let us first consider a dispersion law such that  $\psi(k) = \gamma k^2$  (with  $\gamma > 0$ ), meaning that 3-ph processes are allowed for any value of the momentum of the interacting particles. By a power counting analysis one readily deduces the scaling  $\eta \propto 1/T^5$ , which is the same scaling obtained with binary collisions of phonons [40, 42], and from the expressions derived above we obtain

$$\eta_{3\text{ph}} \simeq 2.1 \times 10^{-7} \frac{c_s^6 \rho}{T^5 \gamma^2 (1+u)^2}. \quad (34)$$



Neglecting  $\mathcal{O}(\psi)$  corrections, the entropy due to the phonons is given by

$$s_{\text{ph}} = \frac{2\pi^2 T^3}{45 c_s^3}, \quad (35)$$

and assuming the values  $\xi = 0.37$ , and that  $|c_1 + 3c_2/2| \sim 0.021$ , we obtain

$$\frac{\eta_{3\text{ph}}}{s_{\text{ph}}} \simeq 4.0 \times 10^{-9} \frac{T_F^8}{T^8}. \quad (36)$$

This should be compared with the shear viscosity coefficient obtained considering binary collisions of phonons. At the LO in the momentum expansion [1, 40, 42]

$$\frac{\eta_{4\text{ph}}}{s_{\text{ph}}} \simeq 2.2 \cdot 10^{-7} \frac{T_F^8}{T^8}, \quad (37)$$

which is about two orders of magnitude larger than  $\eta_{3\text{ph}}$ . Also notice that  $\eta_{4\text{ph}}$  does not diverge for  $\gamma = 0$  because no kinematical restriction applies to 4-ph processes.

At this point it is important to remark that the above expressions lead to two unphysical results. Both  $\eta_{3\text{ph}}$  and  $\eta_{4\text{ph}}$  diverge at vanishing temperature, and both are below the experimental values (and below the universal limit) for sufficiently high temperatures. We shall see that these behaviors are due to an oversimplified description of the phonon fluid adopted so far, and they can actually be used to have information about the phonon fluid. We shall discuss the  $T \rightarrow 0$  limit in Sec. IV, and we shall see that the behavior of the shear viscosity in this limit is regulated by the geometrical extension of the trap.

The decrease of  $\eta_{4\text{ph}}/s$  with increasing temperature is due to the fact that a linear dispersion law was used in its computation [40, 42], while this behavior should be modified when considering higher order terms in the phonon dispersion law. Given that  $\gamma \propto 1/k_F^2$  and treating  $\gamma T^2/c_s^2$  as a perturbative parameter, we expect that the first NLO correction to  $\eta_{4\text{ph}}$  leads to

$$\frac{\eta_{4\text{ph}}}{s_{\text{ph}}} \simeq 2.2 \cdot 10^{-7} \frac{T_F^8}{T^8} + B \frac{T_F^6}{T^6}, \quad (38)$$

where  $B$  is some coefficient that depends linearly on  $\gamma$ , and thus vanishes for  $\gamma = 0$ . The correction becomes more and more important when approaching  $T_c$  and moreover with increasing temperature we expect that more terms should be included in the expansion of  $\eta_{4\text{ph}}$  in Eq.(38), eventually preventing  $\eta_{4\text{ph}}/s_{\text{ph}}$  to decrease below the universal bound. Unfortunately, the evaluation of the coefficient  $B$  and of the other corrections is rather complicated. For this reason we shall not include in our discussion the contribution of 4-ph processes.

The decrease of  $\eta_{3\text{ph}}/s$  with increasing temperature is due to the fact we have assumed a dispersion law with  $\psi(k) = \gamma k^2$ . Since the typical momentum of phonons is of order  $10 T/c_s$ , by increasing the temperature, the neglected terms in the  $\psi(k)$  expansion may become relevant. This motivates us to include terms of order  $k^4/k_F^4$  in the expansion of Eq. (11), that is, we take into account the next to next to leading order (NNLO) term in the momentum expansion of the dispersion law. While there are different computations in the literature of  $\gamma$ , the value of  $\delta$  has not been computed yet. We only know that  $\delta \propto 1/k_F^4$ . We assume that  $\delta < 0$ , as the opposite choice leads to even higher values of the viscosity at low  $T$  than those obtained in Eq. (34). In the following section we shall determine the approximate value of  $\delta$  from a fit of the experimental data.

With this choice of the sign of  $\delta$ , the function  $\psi$  is only positive in a restricted region of momenta. This has as a direct consequence that the three phonon processes are only possible when the phonon momenta are in a certain range of values. The constraints of energy and momentum conservation at the 3-ph vertex, implies that the integration ranges in the integral in Eq. (33) are given by  $0 \leq k' \leq \sqrt{\frac{6}{5\beta}}$ , where  $\beta = -\frac{2\delta}{\gamma}$ , and by  $p'_1 \leq p' \leq p'_2$ , where  $p'_{1,2} = (-k' \pm \sqrt{-3k'^2 + \frac{24}{5\beta}})/2$ . After defining the dimensionless parameter

$$\psi_{\text{max}} = -\frac{\gamma^2}{4\delta}, \quad (39)$$

and

$$\tilde{\beta} = -\frac{2\delta}{\gamma} \frac{T^2}{c_s^2}, \quad (40)$$

we can express the shear viscosity coefficient as

$$\eta_{3\text{ph}} = \frac{2^6 \pi^7}{3^4 5^3} \frac{c_s^2 \rho}{T(1+u)^2} \frac{1}{\psi_{\text{max}}^2 I(\tilde{\beta})}, \quad (41)$$

where

$$I(\tilde{\beta}) = \tilde{\beta}^2 \int^* dx dy (xy(x+y))^4 (6 - 5\tilde{\beta}(x^2 + y^2 + xy))^2 f_{x+y}(1+f_x)(1+f_y), \quad (42)$$

and  $x = p'c_s/T$  and  $y = k'c_s/T$ , so that the integral is done over adimensional variables. This leads to the following expression of the shear viscosity to entropy ratio (we take  $\xi = 0.37$ )

$$\frac{\eta_{3\text{ph}}}{s_{\text{ph}}} \simeq \frac{0.09}{\psi_{\text{max}}^2} \frac{1}{I(\tilde{\beta})} \frac{T_F^4}{T^4}. \quad (43)$$

The quantity  $\tilde{\beta}$  depends on  $T$ , and therefore the temperature dependence of  $\eta_{3\text{ph}}/s$  is complicated. However, we find that for  $0.003 \lesssim \tilde{\beta} \lesssim 0.015$ ,  $I(\tilde{\beta}) \approx 2046.4$ , independent of  $T$ . In this region, we thus obtain that  $\eta_{3\text{ph}} \propto 1/T$ , and then

$$\frac{\eta_{3\text{ph}}}{s_{\text{ph}}} \simeq \frac{4.2 \cdot 10^{-5}}{\psi_{\text{max}}^2} \frac{T_F^4}{T^4} \quad \text{for } 0.05 T_F \lesssim T \lesssim 0.12 T_F, \quad (44)$$

where the temperature range has been evaluated considering  $\psi_{\text{max}} \sim 0.2 - 0.4$ , see Sec. IV B.

For smaller values of the temperature we obtain that  $I(\tilde{\beta}) \propto \tilde{\beta}^2$ , leading to a viscosity  $\eta_{3\text{ph}} \propto 1/T^5$ , in agreement with the results obtained using a vanishing  $\delta$ . This is as expected, because the leading contribution to  $\eta_{3\text{ph}}$  comes from phonons with momenta of the order  $10T/c_s$ , and at very low temperature, the correction introduced by the  $\delta$  term of the dispersion law is of the order  $T^4/(c_s k_F)^4$ , and becomes negligible. We have not obtained an analytical expression of  $I(\tilde{\beta})$  for arbitrary values of  $\tilde{\beta}$ , thus we will present only numerical results in Sec. IV for particular values of  $\psi_{\text{max}}$ .

From these results it is apparent that the 3-ph contribution to the shear viscosity coefficient is extremely sensitive to the dispersion law of phonons. At the LO (neglecting terms proportional to  $\gamma$  and  $\delta$ ), the shear viscosity coefficient diverges (see the discussion after Eq.(13)). Including the NLO term proportional to  $\gamma$  leads to  $\eta_{3\text{ph}} \propto 1/T^5$ , and finally including the NNLO term proportional to  $\delta$  gives, in a certain temperature range,  $\eta_{3\text{ph}} \propto 1/T$ . These results have to be contrasted with the effect of the neglected terms in the vertices of interactions. These terms would lead to corrections to the shear viscosity coefficient proportional to  $(T/T_F)^2$  (or higher powers) and are therefore negligible. In other words, the dispersion law determines the leading temperature dependence of  $\eta_{3\text{ph}}$ , the vertex corrections lead to sub-leading temperature dependence.

Given the strong dependence of the shear viscosity coefficient on the phonon dispersion law, one may question whether with increasing temperature higher order terms in the phonon Lagrangian may be relevant. In the Appendix A we argue that the neglected terms should be relevant only for temperatures  $T \sim (0.2 - 0.3)T_F$ ; given that the transition to the normal phase takes place at  $T_c \simeq 0.16T_F$ <sup>2</sup> [44, 45], those terms are irrelevant for a discussion of the shear viscosity coefficient in the superfluid phase.

#### IV. SHEAR VISCOSITY IN TRAPPED FERMIONIC GASES AND COMPARISON WITH THE EXPERIMENTAL VALUES

The experimental measurement of the shear viscosity of ultracold fermionic atoms is performed confining a degenerate mixture of an equal number of spin-1/2-up and spin-1/2-down in an optical trap. In the following we shall refer to the experimental setup realized in Refs. [20, 22] with  $N \simeq 2.0 \times 10^5$   $^6\text{Li}$  atoms in a magnetic field of strength  $B = 840$  G, close to the  $s$ -channel Feshbach resonance at  $B = 834.15$  G [46]. The unperturbed number density of the trapped fermions is given by

$$\rho_0(\mathbf{r}) = \frac{(2mE_F)^{3/2}}{3\pi^2} \left( 1 - \sum_{i=1}^3 \frac{r_i^2}{R_i^2} \right)^{3/2} \quad R_i = \sqrt{\frac{2E_F}{m\omega_i^2}}, \quad (45)$$

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<sup>2</sup> This is the value of the critical temperature in an homogenous system

where  $m$  is the mass of the  ${}^6\text{Li}$  atom and  $E_F$  is the Fermi energy of  $N$  free fermions in a harmonic oscillator potential. In Refs. [20, 22], the transverse frequencies of the trap are  $\omega_x \simeq 2\pi \times 1778 \text{ s}^{-1}$ ,  $\omega_y \simeq 2\pi \times 1617 \text{ s}^{-1}$  and the net axial frequency is  $\omega_z \simeq 2\pi \times 76 \text{ s}^{-1}$ , corresponding to a Fermi temperature  $T_F \simeq 2.4 \cdot 10^{-6} \text{ K}$ . Given the asymmetry of the trap, it is convenient to redefine the radial coordinate as

$$\omega^2 r^2 = \omega_x x^2 + \omega_y y^2 + \omega_z z^2 \quad (46)$$

where  $\omega = (\omega_x \omega_y \omega_z)^{1/3}$ . Using this definition of the radial coordinate, the system is spherically symmetric, indeed the trapping potential, defined in Eq. (9), is given by  $V(r) = \frac{1}{2} \omega^2 r^2$ . The shear viscosity can be extracted at both high and low temperatures by studying either the anisotropic expansion of the atomic cloud or the breathing mode damping [47].

The critical temperature of the trapped system is  $T_c^{\text{trap}} \simeq (0.21 - 0.25)T_F$  [21, 22], and corresponds to the temperature for which no region of the system is in the superfluid phase. Whether the system is locally in the superfluid phase depends on the local value of the Fermi temperature, and it is found that the system is superfluid for  $T < 0.16T_F(r)$ . Here  $T_F(r)$  is the local value of the Fermi temperature in the LDA. The part of the fermionic cloud which is superfluid occupies the trap center, that extends up to a distance  $\bar{r}(T)$  such that  $T = 0.16T_F(\bar{r})$ . Since at very low temperature the matter distribution in the trap center is roughly constant, and the largest part of the fluid is in the trap center, we shall assume that  $T_F(r) = T_F$  for  $r \leq \bar{r}$ . The trap center is surrounded by a corona of normal fluid, which corresponds to the range  $r > \bar{r}$ , where  $T > 0.16T_F(r)$ . For  $T > T_c^{\text{trap}}$  one has that  $\bar{r} = 0$ ; only fermionic modes are present and their collision processes determine the value of the shear viscosity coefficient. At high  $T$  a simple dimensional analysis, or a much careful study based on a Boltzmann equation, predicts the scaling  $\eta \propto T^{3/2}$  [48], in agreement with the experimental results [47]. For a recent determination of the shear viscosity using quantum Monte Carlo calculations see Refs. [49, 50].

For temperatures below the superfluid phase transition, both the theoretical and experimental studies of the transport properties of the system are quite challenging. In this case the low energy spectrum of the system consists of two types of quasiparticles: the fermions and the phonons. The phonons are present in the center of the trap, where the superfluid phase is realized, while the fermionic modes are present in the trap center as well as in the corona. We shall assume that the leading contribution to the shear viscosity is due to the trap center, where the density is larger. In the trap center the system is superfluid and therefore the quasiparticle fermions are gapped, and their density is exponentially suppressed as  $e^{-\Delta/T}$ , where  $\Delta$  is the value of the fermionic gap. Nevertheless, their contribution to the shear viscosity may be sizable at any temperature. Indeed it was shown by Pethick *et al.* [51] that the shear viscosity coefficient of superfluid  ${}^3\text{He}$  is not exponentially suppressed at low temperature, but tends to a constant value  $\sim 1/4\eta(T_c)$ . The case of trapped ultracold fermionic system close to unitarity was studied in a recent publication [52] and the contribution of fermionic modes to the shear viscosity was analyzed by computing the value of the effective carrier fermion number in the superfluid phase as a function of  $T$ . It has been concluded that the fermionic modes in the superfluid phase cannot be responsible for the experimental values of the shear viscosity below  $T_c$ . Instead, it is suggested that the shear viscosity is dominated by the fermions in the corona of the trap. However, the obtained numerical values of  $\eta/s$  are about a factor two larger than the experimentally measured values.

Since phonons are gapless excitations, their number density is not exponentially suppressed at low  $T$ , but decreases as  $T^3$ . Therefore, at sufficiently low temperature they should give a sizable contribution to the thermodynamic quantities as well as to the transport properties of ultracold fermionic systems. In the BEC limit it is indeed known that below  $T_c$ , phonons give the dominant contribution to the entropy of the system as well to the shear viscosity coefficient [12]. On the other hand, in the BCS limit their contribution is negligible as compared to the fermionic contribution. At unitarity the contributions of fermionic and bosonic modes to the thermodynamic quantities might be comparable, as shown in [30] studying gaussian fluctuations on the top of a mean field solution. However, it is reasonable to expect that at sufficiently low temperature the contribution of fermions to the entropy of the system will be exponentially suppressed. Therefore, we shall assume that  $s = s_{\text{ph}}$ , given by the expression in Eq. (35).

Regarding the shear viscosity coefficient, one has to combine the fermionic and the bosonic contributions. One possible and challenging way of doing this, is to solve the Boltzmann equations which take into account both degrees of freedom. However, since the results of Ref. [52] show that the fermionic contribution to  $\eta$  is above the experimental values, there should be a mode with a lower shear viscosity coefficient. Indeed,  $\eta$  is directly proportional to the typical time for transport of momentum in the direction orthogonal to the flow,  $\tau_{\perp}$ , and therefore a mode with a shorter  $\tau_{\perp}$  should exist. We assume that this mode is the phonon. Actually, the authors of Ref. [52] claim that the interaction between phonons cannot lead to the experimentally measured value of the viscosity, because phonons do not couple to transverse probes. They also notice that the contribution to  $\eta$  found in Ref. [42] increases when the temperature is lowered, in disagreement with the experimental results. However, the arguments presented in Ref. [52] are not conclusive. As we shall show in the next section in the analysis of the low temperature regime, the inclusion of finite size effects in the phonon dynamics leads to a contribution to  $\eta$  that decreases when the temperature decreases. Furthermore, there is no physical reason why a quasiparticle of the system that transports energy cannot

transport momentum in the direction orthogonal to the direction of the flow. In  $^4\text{He}$ , indeed, it has been shown that at sufficiently low temperature phonons give a sizable contribution to the shear viscosity [12], and an analogous result should hold for any ultracold superfluid.

When the temperature of the system approaches  $T_c^{\text{trap}}$  from below, the contribution of fermions to the shear viscosity and to the entropy of the system should increase, for two reasons. In the first place, the radius of the trap center decreases, and a larger portion of the system is in the normal phase. Secondly, the value of the quasiparticle gap  $\Delta$  should decrease with increasing temperature and therefore close to  $T_c^{\text{trap}}$  the fermionic modes in the trap center should not be exponentially suppressed. Indeed, close to  $T_c^{\text{trap}}$  the thermodynamic description in terms of fermionic modes gives an excellent description of the experimentally measured properties of the system [21, 34]. On the other hand, starting from vanishing temperature and increasing  $T$ , both effects mentioned above result in an increasing importance of the fermionic contributions, however it is not obvious at which temperature the fermionic contribution will dominate the bosonic one. At the present stage, only a comparison with experimental results can help to determine such a temperature. In the following we shall discuss the contribution of the bosonic modes to  $\eta/s$ , and show that the very low temperature experimental points can be described by taking into account solely the contribution of phonons.

### A. Finite size effects

As already mentioned in Sec. IIIB, the contributions to the shear viscosity of 4-ph and 3-ph processes diverge at vanishing temperature. This unphysical behavior is due to the fact we have performed the various calculations in an infinite volume. In order to properly evaluate the contribution of phonons to the shear viscosity, it is important to consider the finite extension of the optical trap and the corresponding matter distribution. In the LDA the effect of the matter distribution can be taken into account computing the trap average of the shear viscosity evaluated in the previous section. At very low temperature, the matter distribution in the trap center is roughly constant, indeed we have assumed  $T_F(r) = T_F$  for  $r \leq \bar{r}$ , and we expect that the trap average would lead to a tiny variation of our results.

However, the finite extension of the trap does play an important role because the interaction of quasiparticles with the boundary can be neglected only if their mean free path is sufficiently small. More precisely, finite size effects have to be taken into account when the Knudsen number  $K_n = \frac{l}{L}$ , defined as the ratio of the mean free path  $l$  of the quasiparticles over the typical size of the system  $L$ , is not small.

The phonon mean free path  $l_{\text{ph}}$  can be obtained by considering the same 3-ph and 4-ph processes we have taken into account for the evaluation of the shear viscosity. The shear viscosity and the corresponding mean free path can be readily evaluated by the formula

$$\eta_{\text{bulk}} = \frac{1}{5} \rho_{\text{ph}} c_s l_{\text{ph}}, \quad (47)$$

where  $\rho_{\text{ph}} = \frac{2\pi^2 T^4}{45c_s^3}$  is the phonon density of the normal fluid component. At low  $T$ , for the 3-ph processes one obtains that  $l_{3\text{ph}} \propto 1/T^5$ , while for 4-ph collisions  $l_{4\text{ph}} \propto 1/T^9$ . Thus, no matter which process dominates, for sufficiently low temperature the mean free path becomes larger than the typical length of the system. In the case we are interested in, we take as the typical size the smallest radius of the atomic cloud,  $R_x$ , given in Eq. (45). We present in Fig.1 a plot of the value of  $l_{\text{ph}}/R_x$  as a function of  $T/T_F$  for both 3-ph and 4-ph processes. The vertical dashed green line in this plot and in all the subsequent plots indicates the critical temperature  $T_c^{\text{trap}} = 0.25T_F$  reported in [22]. However, the critical temperature might be slightly lower, i.e.  $T_c^{\text{trap}} \simeq 0.21T_F$ , as obtained in [21]. It has to be understood that the various plots of the phonon mean free paths are meaningful only for values of  $T < T_c^{\text{trap}}$  because phonons do not exist in the normal phase. For the same reason, any plot of the phonon shear viscosity coefficient is meaningful only in the superfluid phase. Regarding the 3-ph processes the mean free path strongly depends on the value of  $\psi_{\text{max}}$ . As we shall see in the next section, the experimental data seem to favor the values  $\psi_{\text{max}} \sim 0.2 - 0.3$ , and the lines named 3ph-a, 3ph-b and 3ph-c correspond to the mean free paths obtained with  $\psi_{\text{max}} = 0.4$ ,  $\psi_{\text{max}} = 0.3$  and  $\psi_{\text{max}} = 0.2$  respectively. For  $\psi_{\text{max}} \lesssim 0.2$  the mean free path of the 3-ph process is always larger than  $R_x$ . For  $\psi_{\text{max}} > 0.3$  the mean free path of the 3-ph process becomes of the order of the size of the atomic cloud only for  $T \sim 0.12T_F$ , and rapidly increases for lower values of  $T$ . We also present the plot of  $l_{4\text{ph}}/R_x$ , (named 4-ph) corresponding to the dashed black line in Fig.1. Also in this case the mean free path becomes of the order of the trap radius  $R_x$  for  $T \sim 0.12T_F$ . Notice that the curves 3-ph-a and 4-ph almost overlap for  $T \lesssim 0.15T_F$ , however for larger temperatures 4-ph becomes extremely small, while in the 3-ph-a case it remains of the order of  $0.1R_x$ . Such a difference is due to the fact that the 3-ph shear viscosity has been evaluated including the term proportional to  $\psi(k)$  in the dispersion law of phonons. In this case the phonon dispersion law has a maximum, which leads to a nonzero minimum of the corresponding mean free path for a certain value of  $T$ . The  $\psi(k)$  term was not included in the evaluation of the 4-ph shear viscosity and for this reason,

the 4-ph curve of the mean free path does not have a minimum and should not be very reliable when getting close to  $T_c^{\text{trap}}$ . As a check we have evaluated the mean free path for the 3-ph processes considering  $\psi_{\text{max}} \gg 1$  (corresponding to  $\delta \ll \gamma^2$ , see Eq.(39)), meaning that the effect of the maximum in the phonon dispersion law manifests at very high momenta (and temperature). In the temperature range reported in Fig. 1 we have obtained a plot of the mean free path in concordance with the results that obtained for  $\delta = 0$  and very similar to the one of the 4-ph processes.

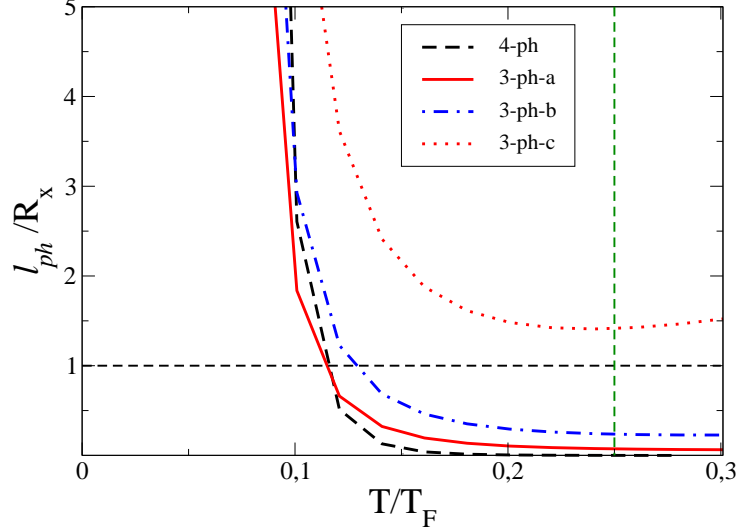


FIG. 1: Plot of the Knudsen number,  $l_{\text{ph}}/R_x$ , as a function of  $T/T_F$  for 3-phonon (3-ph) and 4-phonon (4-ph) processes. The black dashed line corresponds to the mean free path associated to the 4-ph process. The solid red line (named 3-ph-a) corresponds to the 3-ph process with  $\psi_{\text{max}} = 0.4$ , the dot-dashed blue line (named 3-ph-b) corresponds to the 3-ph process with  $\psi_{\text{max}} = 0.3$  and the dotted red line (named 3-ph-c) corresponds to the 3-ph process with  $\psi_{\text{max}} = 0.2$ . The horizontal black dashed line corresponds to  $l_{\text{ph}}/R_x = 1$ . The hydrodynamic description is valid for  $l_{\text{ph}}/R_x \ll 1$ . The vertical dashed green line approximately corresponds to the transition temperature between the normal phase and the superfluid phase [22].

When the mean free path of the phonons exceeds the typical size of the superfluid region, the transport properties of the phonons are mainly governed by their interactions with the boundary, rather than by the self-interactions. In the trapped atomic system the density is not constant, and the superfluid region at the trap center is surrounded by a corona of a dilute fluid of unpaired fermions. This layer effectively acts as a boundary for the superfluid phonons, which are confined to the superfluid center of the cloud, and when phonons reach this region they are absorbed or scattered back. If the process is elastic, then the phonon will only exert a normal pressure on the boundary, resulting in the propagation of low energy excitations in the normal phase corresponding to standard first sound. But, if the phonon is diffused, then it will as well exert a shear stress on the boundary inasmuch as the particles of a rarefied gas produce a shear stress when diffused by the walls of a container.

In principle, both kind of processes, self-interactions and interactions with the boundary, take place for any value of the Knudsen number. However in the hydrodynamic regime, for  $K_n \ll 1$ , self-interactions are the dominant microscopic processes leading to the largest contribution to the transport coefficients, while the interactions with the boundary are only relevant in a layer of length  $l_{\text{ph}}$  (the Knudsen layer) close to the boundary. The interaction with the boundary can be effectively taken into account in the hydrodynamic equations by requiring that the fluid velocity vanishes at the boundary [23–25], the so-called “no-slip” boundary condition.

Finite size effects in the flow of rarefied gases and fluids have been studied thoroughly in the past [23–25]. Depending on the value of  $K_n$  they can be taken care of either by imposing a “slip” boundary condition in the solution of the Navier-Stokes equations, or, at a microscopic level, by using the Boltzmann equation with appropriate boundary conditions. For values of  $K_n \lesssim 0.1$  (say) the Navier-Stokes equation are still valid, but the shear viscosity coefficient (and the transport coefficients in general) must be corrected including the finite size corrections, which for low Knudsen numbers can be expressed as

$$\frac{1}{\eta_{\text{slip}}} = \frac{1}{\eta_{\text{bulk}}} \left( 1 + \frac{\zeta l}{L} \right), \quad (48)$$



where  $\eta_{\text{bulk}}$  refers to the value of the viscosity in the hydrodynamic regime, and  $\zeta$  is a coefficient that characterizes the interaction of the quasiparticles with the boundaries for a particular geometry of the system. Eq. (48) gives account to a correction to the standard hydrodynamic equations proportional to  $K_n$ . With increasing  $l$ , terms of higher order in  $K_n$  must be added, but the computation of such terms is complicated even for an ideal gas [23, 24]. In any case, with increasing values of  $K_n$  one is forced to abandon the Navier-Stokes equation and more moments of the distribution function have to be considered. For recent progress with Grad's moment method see e.g. [25].

For values of  $K_n \gg 1$ , the system is in the ballistic (also named Knudsen) regime and no hydrodynamic description can be employed for the description of the phonon gas. In this case the interactions of phonons with the boundary are more frequent than the interactions among phonons, and the relevant relaxation time is the time between two collisions of a particle with the boundary,  $\tau_b$ . As an admittedly rough model of the interaction between the phonon fluid and the fermions in the external corona, we assume that the boundary between the superfluid and the normal region is a sharp surface, i.e. a *simple boundary* [24], and, as in the Maxwell model [24, 25], the scattering at the boundary is described by one single parameter, the *accommodation coefficient*  $\chi$ , meaning that particles impinging on the boundary are diffused with probability  $\chi$  and specular-reflected with probability  $1 - \chi$ . This simplified description allows us to qualitatively capture the physical process happening at the interface, where a shear stress is exerted by the phonon gas on the fermionic fluid of the external corona. A more detailed and realistic modeling of the interaction of phonons with the normal fermions is certainly possible, however such a description would require the knowledge of the matter distribution, as well as the computation of the interaction of phonons with (almost) gapless fermionic modes. Moreover, for a proper description of the interface between the normal and superfluid component one should take into account the transfer of matter between the two phases. This effect can be taken into account by introducing one further phenomenological parameter, usually named the *condensation coefficient* [24], which is related to matter transfer between the two phases. We postpone a detailed analysis of the interface to future work and in the present paper we content ourselves with a phenomenological formula for the shear viscosity coefficient which has been used, derived and tested for the description of finite size effects in the flow of rarefied gases and fluids, see e.g. [53], in the presence of a simple boundary.

As in a rarefied gas, the phonons will exert on the boundary a shear stress which is proportional to the energy density of phonons and will damp the oscillation of the boundary. The same phenomenon takes place in  $^4\text{He}$  at extremely low temperature,  $T \lesssim 0.5$  K, where it has been observed that ballistic phonons can efficiently damp the movement of immersed objects. Experiments done with an oscillating sphere [54], a vibrating microsphere [55], or a vibrating quartz tuning fork [41] immersed in superfluid  $^4\text{He}$  (see [41] also for a comparison of the results obtained with various experimental apparatuses) show that the damping of the oscillations can be described, in perfect accommodation ( $\chi = 1$ ), by the introduction of a ballistic (effective) shear viscosity coefficient defined by

$$\eta_{\text{ball}} \equiv \frac{1}{5} \rho_{\text{ph}} c_s d, \quad (49)$$

where  $d$  is the typical size of the oscillating object. This expression leads to excellent agreement with the experimental data for large Knudsen numbers [41], predicting the correct dependence on the temperature and on the typical size of the oscillating object.

Note that the expression above is equivalent to the one in Eq.(47) with the replacement of  $l_{\text{ph}}$  with  $d$  and suggests that in the ballistic regime (and in the presence of a boundary) one can extend the standard definition of the transport coefficient by replacing the mean free path with the typical size of the system. Therefore, in our case, we define a ballistic (effective) shear viscosity as

$$\eta_{\text{ball}} = \frac{1}{5} \rho_{\text{ph}} c_s \chi (c_s \tau_b) \equiv \frac{1}{5} \rho_{\text{ph}} c_s a, \quad (50)$$

where  $c_s \tau_b = L$  is the typical size of the system, i.e. the extension of the superfluid region at the trap center (which is smaller than  $R_x$ ). In the present paper we are not interested in a detailed description of the interface between the normal and the superfluid phase but rather to understand whether phonons can give a sizable contribution to the shear viscosity coefficient. For this reason, we define the phenomenological length  $a = \chi L$ , which we will use as a fitting parameter.

In order to evaluate the shear viscosity in the intermediate region,  $K_n \sim 1$ , one should employ the Boltzmann equation treating collisions of phonons with the boundary and collisions among phonons on an equal footing. Solving such a problem is complicated even for ideal gases [23–25]. However, since we know the behavior of the relaxation time for large values of  $K_n$  as well as for  $K_n \ll 1$ , it is possible to employ the same reasoning at the basis of the Matthiessen's rule [56] to obtain an expression of  $\eta$  in the intermediate region, which interpolates between the values of the shear viscosity coefficient in the Knudsen and in the hydrodynamic regimes. For this purpose we define an effective relaxation time,  $\tau$ , incorporating the effects of inter-particle collisions and of the collisions with the boundary

$$\tau^{-1} = \tau_b^{-1} + \tau_{\text{ph}}^{-1}. \quad (51)$$



This relation follows from the assumption that the total collision frequency is the sum of the frequencies of the two mentioned kinds of collisions, thus we are assuming that the two collision processes are not correlated. Since the shear viscosity coefficient is proportional to the collision time, we define the total effective shear viscosity as

$$\eta_{\text{eff}} = \left( \eta_{3\text{ph}}^{-1} + \eta_{\text{ball}}^{-1} \right)^{-1}, \quad (52)$$

where  $\eta_{\text{ball}}$  is defined in Eq.(50) and  $\eta_{3\text{ph}}$  is the 3ph shear viscosity. In principle, the contribution of the 4ph collisions should be considered as well. However, the  $\eta_{4\text{ph}}$  has been computed assuming  $\psi(k) = 0$ . This approximation should not be of great impact in the evaluation of the shear viscosity at low temperature as shown in Refs. [40, 42]. However, it affects the higher temperature behavior of  $\eta_{4\text{ph}}$ , and for this reason it cannot be included consistently in our analysis. That is why we will neglect binary collisions in the following subsection.

## B. Comparison with the experimental data

The experimental values of the shear to entropy ratio have been obtained by the Duke group in a number of different settings. In the present paper we refer to the data set reported in Ref. [47] and we show that superfluid phonons can explain the low  $T$  experimental behavior of the shear viscosity coefficient. The caveat is that in our analysis we use several simplifying assumptions and two fitting parameters. Let us summarize our assumptions. We consider that the only contributions to the shear viscosity are due to processes taking place among phonons or between phonons and fermions at the interface between the superfluid and the normal phase. We neglect, instead, contributions to  $\eta$  due to interactions between the fermions. In other words we assume that any oscillation of the corona is damped by the friction with the trap center, and this process is effectively described by the ballistic shear viscosity in Eq. (50). With increasing temperature, the contribution to  $\eta$  of the interaction processes among fermions should become more and more relevant due to the fact that the spatial extension and the density of the external corona increases and because more Bogolyubov modes are available. Therefore our results should be reliable only up to some temperature  $T < T_c$ .

Regarding the phonon contribution, we also neglect corrections to the shear viscosity coefficient coming from the higher energy part of the phonon spectrum. This amounts to neglect terms of order  $(k/k_F)^7$  in the phonon dispersion law. In the Appendix A we show that these terms should be negligible for temperatures up to  $(0.2 - 0.3)T_F$ .

While our formalism allows us to compute the shear viscosity in any point of the trap, performing the trap average is not a simple task. Thus, we assume that the largest contribution comes from the trap center, where the density is larger and almost constant, and where the local Fermi temperature,  $T_F(r)$ , is roughly constant and we approximate it with the global value  $T_F$ .

In order to explain the low  $T$  data we will use Eq. (52), which depends on two unknown parameters,  $a$  and  $\delta$  (or equivalently  $\psi_{\text{max}}$ ). We remind that the value of  $\delta$  could be in principle determined if the phonon dispersion law is computed to next to next to leading order, while  $a$  requires to know both the size of the superfluid core in the trap and the probability of the phonons to be diffused at the edges of the core.

In Fig. 2 the experimental values of  $\eta/s$  obtained in [47] are shown, together with three fits obtained by considering the contribution of the interaction of phonons with the boundary, described by Eq.(50), assuming  $a = 0.3 R_x$ , and the 3-ph processes with  $\psi_{\text{max}} = 0.2$  (dashed black line),  $\psi_{\text{max}} = 0.3$  (dotted red line) and  $\psi_{\text{max}} = 0.4$  (solid blue line). The experimental data seem to favor the value  $\psi_{\text{max}} = 0.3$ . However, more experimental data are needed in order to figure out the correct value of  $\psi_{\text{max}}$ . Note also that the experimental values of  $\eta/s$  decreases with decreasing temperature and this behavior is well reproduced for  $\psi_{\text{max}} \lesssim 0.3$ . Although all the experimental values are above the value  $1/4\pi$  it seems that further decreasing the temperature values below the universal bound might be reached.

In Fig. 3 we present three fits of the experimental data considering the contribution of the 3-ph processes (with  $\psi_{\text{max}} = 0.3$ ) and the interaction of phonons with the boundary, effectively describe by Eq.(50) with  $a = 0.2 R_x$  (dashed black line),  $a = 0.3 R_x$  (dotted red line) and  $a = 0.4 R_x$  (solid blue line). The experimental data seem to favor  $a = 0.3 R_x$ , but more experimental data are needed to determine the precise value of this parameter.

In Fig.4 we report several plots of all the various contributions separately as well as the effective shear viscosity. The interaction of the phonons with the boundary (dashed black line) is obtained from Eq. (50) assuming  $a = 0.3 R_x$ ; the contribution of the 3-ph process (dotted red line) is obtained from Eq. (43) considering  $\psi_{\text{max}} = 0.3$ ; the contribution of the 4-ph process (dot dashed green line) is obtained from Eq. (37). The effective shear viscosity (solid blue line) is obtained combining  $\eta_{\text{ball}}$  and  $\eta_{3\text{ph}}$  in Eq. (52) with the above reported values of  $a$  and  $\psi_{\text{max}}$ . With this choice of the parameters the experimental values are appropriately reproduced.

We refrain from showing results for the effective shear viscosity including 4ph processes. Actually, for sufficiently low temperature the expression of  $\eta_{4\text{ph}}$  reported in Eq. (37) is reliable and it can be included in the computation of the effective shear viscosity; but in this case the contribution of 4ph processes is however negligible, because of the

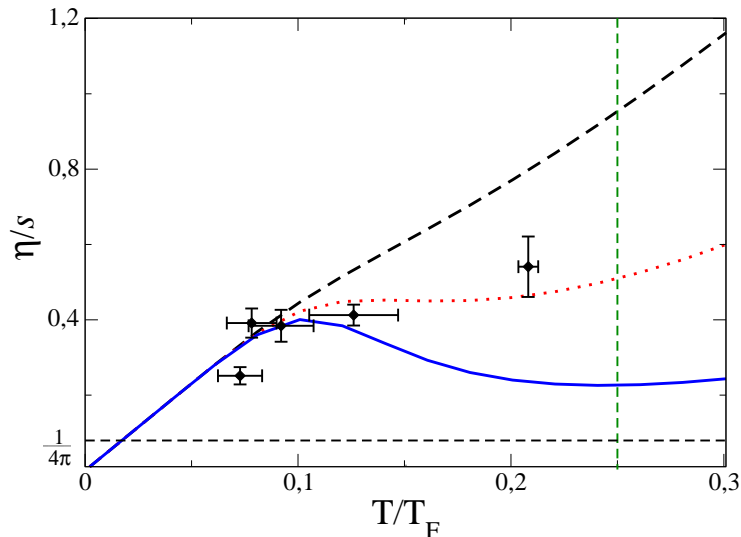


FIG. 2: Plot of  $\eta/s$  in units of  $\hbar/k_B$  as a function of  $T/T_F$  considering the contribution of the interaction with the boundary, assuming  $a = 0.3 R_x$ , and the 3-ph processes with three different values of  $\psi_{\max}$ . The dashed black line is obtained with  $\psi_{\max} = 0.2$ , the dotted red line is obtained with  $\psi_{\max} = 0.3$  and the solid blue line is obtained with  $\psi_{\max} = 0.4$ . The experimental values and the corresponding error bars were taken from [47]. The vertical dashed green line approximately corresponds to the transition temperature between the normal phase and the superfluid phase in a trap reported [22]. The actual value of  $T_c^{\text{trap}}$  might be slightly lower; the value reported in [21] is  $T_c^{\text{trap}} \simeq 0.21 T_F$ .

dominant role of the ballistic viscosity. For higher temperature Eq. (37) is not reliable and NLO contributions of the phonon dispersion law should be included.

Our plots show that at very low temperatures, below  $0.1 T_F$ , the ballistic shear viscosity dominates, meaning that the only relevant dissipative mechanism is the one that takes place at the interface between the normal and the superfluid phase. This result is independent of the detailed expression of the phonon dispersion law, in particular this result remains correct also if 3-ph processes are not allowed, i.e. considering  $\gamma < 0$ , and only binary collisions are included in the analysis. It only depends on the fact that at sufficiently low temperature phonons become ballistic, see Fig. 1.

## V. SUMMARY

Let us give a very brief summary of our results. Below the critical temperature the trapped Fermi gas is in a superfluid phase, characterized by the spontaneous breaking of a  $U(1)$  symmetry. In this regime there are two types of quasiparticles: the fermions and the phonon (Nambu-Goldstone) modes. At sufficiently low  $T$ , only the phonons give relevant contributions to the thermodynamics and transport coefficients of the system, as the fermions are gapped and thermally suppressed. At unitarity, EFT techniques and numerical methods can be used to write down the phonon Lagrangian and to assess the main phonon self-interactions, but this information is still incomplete, because not all the terms in the Lagrangian beyond the NLO have been determined, so far. Further and complementary information can be deduced by the analysis of the experimental data. This mixed approach, already used for different superfluids such as  $^4\text{He}$ , allows us to constrain some parameters of the effective low energy Lagrangian and to understand the most important physical processes.

We have presented a detailed computation of the contribution of small-angle collisions to the shear viscosity. The  $T$  dependence of such a contribution strongly depends on the form of the phonon dispersion law, as it had previously been discovered in  $^4\text{He}$  [14]. In this regard, one notices that the knowledge of the value of  $\eta$  in the  $T$  regime where it is dominated by phonons would certainly provide detailed information on their dispersion law.

We have then discussed the relevance of the restricted geometry for trapped Fermi gases in the evaluation of the shear viscosity. Below  $T_c^{\text{trap}}$  the superfluid phase is realized in the core of the gas cloud, while being surrounded by an outer layer of fermions in the normal phase. At  $T \simeq 0.12 T_F$ , the phonon mean free path becomes of the same

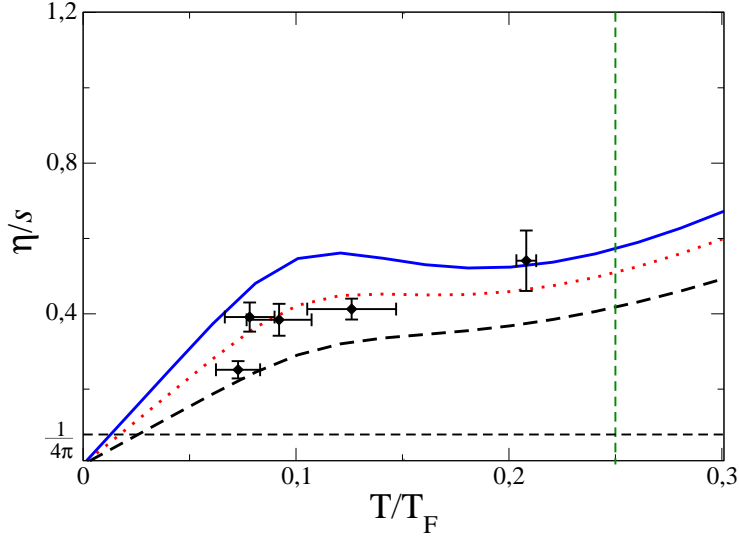


FIG. 3: Plot of  $\eta/s$  in units of  $\hbar/k_B$  as a function of  $T/T_F$  considering the contribution of the interaction with the boundary and the 3-ph processes with  $\psi_{\max} = 0.3$ , and  $a = 0.2 R_x$  (dashed black line),  $a = 0.3 R_x$  (dotted red line) and  $a = 0.4 R_x$  (solid blue line). The experimental values were taken from [47]. The vertical dashed green line approximately corresponds to the transition temperature between the normal phase and the superfluid phase [22].

order, or even larger, than the size of the cloud, and phonons are then in the ballistic regime. Then, phonons collide more often with the boundary of the superfluid core, that is, with the outer layer of normal fermions, than among themselves. Such collisions can provide a shear stress on the boundary, which then results in the damping of the breathing modes. This dissipative mechanism can still be effectively described by a ballistic shear viscosity. The same phenomena occur in  $^4\text{He}$ , where it has been observed that ballistic phonons can damp the movement of immersed objects in the superfluid.

In order to describe the measured values of  $\eta$  for temperatures in between the ballistic regime and the hydrodynamic regime, we have used a phenomenological formula for  $\eta$ , given in Eq. (52), which contains two (yet) unknown parameters, that we fit to the data. One parameter describes the phonon dispersion law at next-to-next-to-leading order, which in principle could be computed from the microscopic physics, while the other one is related to the size of the superfluid core in the trap and the type of phonon-fermion scattering processes that take place at the boundary.

Although our model relies on a number of simplifying assumption, detailed in the previous section, and, as a consequence, gives a rough description of the experimental system, it nevertheless allows us to make a couple of testable semi-quantitative predictions. First, we notice that if experiments are conducted reducing/increasing the size of the trap, but keeping  $E_F$  constant, then  $\eta/s$  at low temperatures should decrease/increase. In other words, the value of  $\eta/s$  should correlate with the size of the gas cloud. Second, we predict that  $\eta/s$  should decrease with decreasing temperature. If we naively extrapolate our results to lower temperature, we predict that there should be a violation of the string theory proposed bound of  $\eta/s$ . Note, however, that this violation happens because phonons are ballistic, while the string theory bound concerns the hydrodynamic regime. Both predictions are independent of the detailed form of the phonon dispersion law, in particular, they are independent of the sign of the  $\gamma$  term in Eq (7). However, if  $\gamma > 0$ , meaning that in a certain range of momenta the Beliaev processes are allowed, it is possible to deduce further information on the phonon dispersion law from the experimental data. In particular, we find that the correction to the dispersion law should be given by the expression reported in Eqs. (10) and (11) with  $\delta \simeq 0.03/k_F^4$ .

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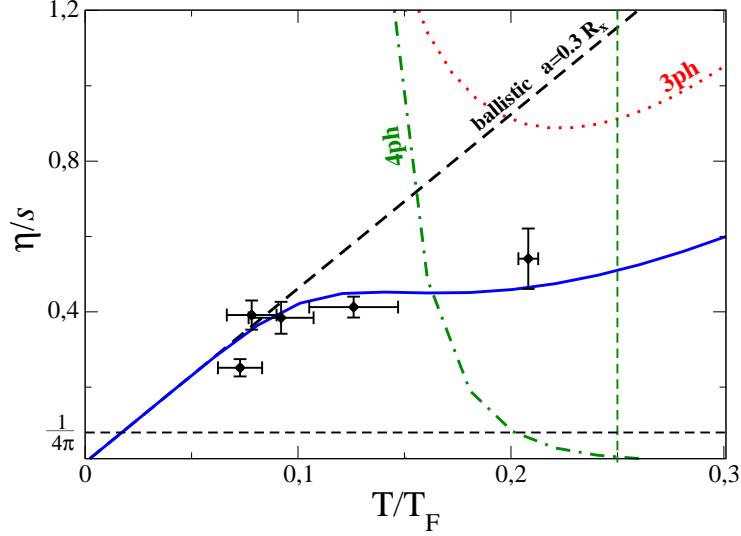


FIG. 4: Plot of  $\eta/s$  in units of  $\hbar/k_B$  as a function of  $T/T_F$ . We report all the various contributions separately as well as the effective shear viscosity obtained with Eq.(52). The interaction of the phonons with the boundary (dashed black line) is obtained from Eq.(50) assuming  $a = 0.3 R_x$ ; the contribution of the 3-ph process (dotted red line) is obtained from Eq.(43) considering  $\psi_{\max} = 0.3$ ; the contribution of the 4-ph process (dot dashed green line) is obtained from Eq.(37). The experimental values and error bars were taken from [47]. The solid blue line is obtained with Eq. (52) combining the ballistic term and 3-ph processes. The vertical dashed green line approximately corresponds to the transition temperature between the normal phase and the superfluid phase [22]. The actual critical temperature may be lower,  $T_c^{\text{trap}} \simeq 0.21T_F$  as reported in [21].

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### Appendix A: Phonon dispersion law and possible “roton” excitations

From the fit of the experimental values of the shear viscosity performed in Sec. IV B we have determined the coefficient  $\delta$  of the phonon dispersion law in Eq.(11). In order to emphasize the region of validity of the expansion of the phonon dispersion law obtained, we rewrite it as an expansion in  $x = k/k_F$ ,

$$E/c_s k_F = x \left( 1 + \tilde{\gamma} x^2 - \frac{\tilde{\gamma}^2}{4\psi_{\max}} x^4 \right), \quad (\text{A1})$$

where we have used Eq.(39), and we have defined  $\gamma = \tilde{\gamma}/k_F^2$ . The corresponding plot, taking  $\psi_{\max} = 0.1$  (solid black line),  $\psi_{\max} = 0.2$  (dashed blue line),  $\psi_{\max} = 0.3$  (dotted red line) is reported in Fig. 5 for positive values of  $E$ .

This plot allows us to figure out whether the neglected terms in the phonon dispersion law are relevant for the present analysis of the shear viscosity coefficient. The terms not included in the momentum expansion are of the order of  $x^7 = (k/k_F)^7$  and one would naively expect that they should give a large contribution to the dispersion law, and to the integral in Eq.(33), as soon as  $k \sim k_F$ . However, the expansion in Eq.(A1) suggests that the actual expansion parameter is not  $x$ , but  $\sqrt{\tilde{\gamma}}x$  and therefore the naive power counting in powers of  $k/k_F$  is incorrect. Assuming that the expansion is in powers of  $\sqrt{\tilde{\gamma}}x$  and considering that  $\tilde{\gamma} \simeq 0.18$ , see Eq.(7), it follows that Eq. (A1) gives a good approximation as far as  $k \lesssim 2k_F$ . This result is corroborated by the observation that the neglected terms should lead to an increase of the phonon dispersion law at  $k \sim (2-3)k_F$  in order to avoid that the dispersion law becomes negative, but should not be relevant at smaller values of  $k$  (otherwise it would lead to the wrong behavior of the shear viscosity at low temperature). Therefore a term, presumably of order  $\tilde{\gamma}^3(k/k_F)^7$  should exist, with positive sign, and lead to a minimum of the dispersion law; such a minimum should correspond to rotons. Regarding the temperature range at which such a term would be relevant, we can estimate it considering that the largest contribution to the shear viscosity coefficient comes from momenta  $k \simeq 10 T/c_s$ . Thus the present analysis is consistent up to temperatures

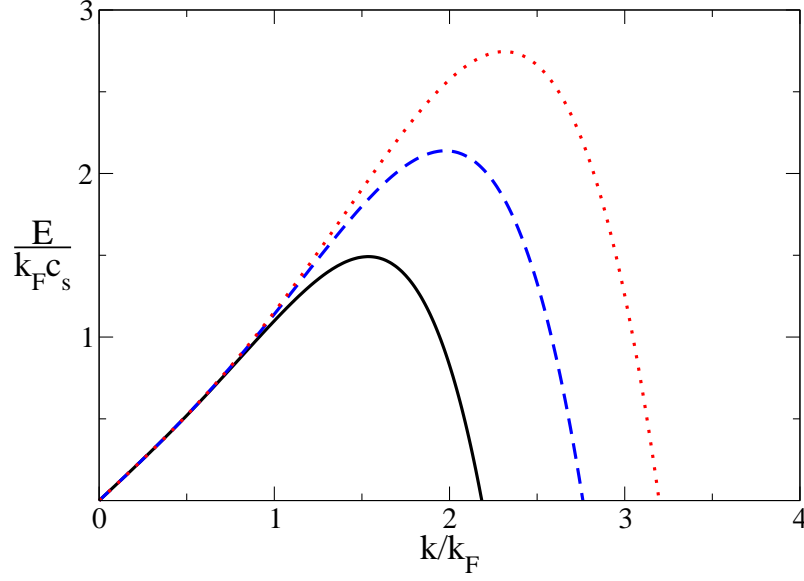


FIG. 5: Plot of the dispersion law of phonons given in Eq. (A1), with  $\psi_{\max} = 0.1$  (solid black line),  $\psi_{\max} = 0.2$  (dashed blue line),  $\psi_{\max} = 0.3$  (dotted red line).

$T \simeq (0.2-0.3) T_F$ , that is for temperatures of the order or larger than the transition temperature. Thus, the neglected part of the spectrum should be important only close to  $T_c$ .

We note that the presence of a dip in the phonon spectrum, both in the BCS and BEC regimes, has been excluded by the study of gaussian quantum fluctuations on the top of a mean field solution [30]. However, the procedure employed in this calculation is only reliable in the BCS and in the BEC limits and it is not clear how a self-consistent calculation might be done close to unitarity (see Section IX of [30]). Moreover, higher order fluctuations of the radial oscillation of the condensate (not included in that analysis) might be relevant close to unitarity, possibly affecting the phonon dispersion law.

### Appendix B: Variational treatment with orthogonal polynomials

In this Appendix we show that with a different choice of family of variational functions we reach to the same value of the shear viscosity as that one found in Sec. III A. Following a treatment very similar to the one carried out in Ref. [42] we select the trial functions as

$$h(p) = p^n \sum_{s=0}^{\infty} b_s B_s(p) , \quad (\text{B1})$$

where  $n$  is a parameter that will be determined by a variational procedure, and  $B_s(p)$  are orthogonal polynomials of order  $s$  and are defined such that the coefficient of the highest power  $p^s$  is 1, and the orthogonality condition

$$\int \frac{d^3p}{(2\pi)^3} \frac{f_{\text{eq}}(1+f_{\text{eq}})}{2E_p} p_{ij} p_{ij} p^n B_r(p) B_s(p) = A_r^{(n)} \delta_{rs} \quad (\text{B2})$$

is satisfied.

Using this form of the solution one can check that Eq. (22) gives

$$\eta_{3\text{ph}} = \frac{2c_s^2}{5T} b_0 A_0^{(n)} , \quad (\text{B3})$$

where

$$A_0^{(n)} = \frac{T^{6+n}}{6\pi^2 c_s^{7+n}} \Gamma(6+n) \zeta(5+n) , \quad (\text{B4})$$

where  $\Gamma(z)$  and  $\zeta(z)$  stand for the Gamma and Riemann zeta functions, respectively.

On the other hand, the expression of the shear viscosity given by Eq. (25) reads

$$\eta_{3\text{ph}} = \sum_{s,t=0}^{N=\infty} b_s b_t M_{st} , \quad (\text{B5})$$

where

$$M_{st} = \frac{1}{5T} \int_{p,p',k'} (2\pi)^4 \delta^{(4)}(P - P' - K') |\mathcal{M}|^2 f_p^{\text{eq}} (1 + f_{p'}^{\text{eq}}) (1 + f_{k'}^{\text{eq}}) \Delta_{ij}^s \Delta_{ij}^t , \quad (\text{B6})$$

and

$$\Delta_{ij}^t = B_t(p) p^n p_{ij} - B_t(p') p'^n p'_{ij} - B_t(k') k'^n k'_{ij} , \quad (\text{B7})$$

Requiring that the two forms of the shear viscosity, Eqs. (B3) and (B5), to be equal implies

$$b_0 = \frac{2c_s^2}{5T} A_0^{(n)} (M^{-1})_{00} , \quad (\text{B8})$$

so that

$$\eta_{3\text{ph}} = \frac{4c_s^4}{25T^2} (A_0^{(n)})^2 (M^{-1})_{00} . \quad (\text{B9})$$

In practical terms one performs the study by limiting the number of orthogonal polynomials included in the analysis. One can prove that

$$\eta_{3\text{ph}} \geq \frac{4c_s^4}{25T^2} (A_0^{(n)})^2 (M^{-1})_{00} , \quad (\text{B10})$$

for a particular value of  $n$  and the number of orthogonal polynomials considered in the study.

For definiteness, we consider  $\psi(p) = \gamma p^2$ . We then find that  $n = -1$  leads to the maximum value of  $\eta_{3\text{ph}}$ . Further, after using only up to the polynomial of order  $s = 2$  we get the convergence to an expression which matches that of Eq. (34).

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- [1] T. Schafer and D. Teaney, Rept. Prog. Phys. **72**, 126001 (2009) [arXiv:0904.3107 [hep-ph]].
  - [2] A. Adams , L.D. Carr, T. Schaefer, P. Steinberg, J.E. Thomas, “Strongly correlated quantum fluids: ultracold quantum gases, quantum chromodynamic plasmas, and holography duality”, [arXiv:1205.5180 [hep-ph]]-
  - [3] S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. **80**, 1215 (2008).
  - [4] T. -L. Ho, Phys. Rev. Lett. **92**, 090402 (2004) [cond-mat/0309109].
  - [5] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade and J. E. Thomas, Science **298**, 2179 (2002) [arXiv:cond-mat/0212463]; M. E. Gehm, S. L. Hemmer, S. R. Granade, K. M. O’Hara and J. E. Thomas, Phys. Rev. A **68**, 011401 (2003).
  - [6] T. Bourdel, J. Cubizolles, L. Khaykovich, K. M. F. Magalhaes, S. J. J. Kokkelmans, G. V. Shlyapnikov and C. Salomon, Phys. Rev. Lett. **91**, 020402 (2003).
  - [7] S. Gupta *et al.*, Science **300**, 1723 (2003).
  - [8] C. A. Regal and D. S. Jin, Phys. Rev. Lett. **90**, 230404 (2003) [arXiv:cond-mat/0302246].
  - [9] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001) [hep-th/0104066].
  - [10] C. Chafin, T. Schafer, [arXiv:1209.1006, [cond-mat.quant-gas]]
  - [11] P. Romatschke, R. E. Young, [arXiv:1209.1604, [cond-mat.quant-gas]]
  - [12] I. M. Khalatnikov, “Introduction to the Theory of Superfluidity”, Benjamin, New York, 1965.
  - [13] Yu.A. Kosevich, Fiz. Nizk. Temp. **9**, 479 (1983) [Low Temp. Phys. **9**, 242 (1983)].
  - [14] H. J. Maris, Phys. Rev. A **8**, 1980 (1973).
  - [15] D. Benin, Phys. Rev. B **11**, 145 (1975).



- [16] D. T. Son and M. Wingate, *Annals Phys.* **321**, 197 (2006) [arXiv:cond-mat/0509786].
- [17] M. Mannarelli, C. Manuel and L. Tolos, [arXiv:1201.4006 [cond-mat.quant-gas]].
- [18] G. Rupak and T. Schafer, *Nucl. Phys. A* **816**, 52 (2009) [arXiv:0804.2678 [nucl-th]].
- [19] L. Salasnich and F. Toigo, *J. Low. Temp. Phys.* **165**, 239 (2011) [arXiv:1107.4552 [cond-mat.quant-gas]].
- [20] J. Kinast, A. Turlapov and J. E. Thomas, *Phys. Rev. Lett.* **94**, 170404 (2005) [arXiv:cond-mat/0502507].
- [21] R. Haussmann and W. Zwerger, *Phys. Rev. A* **78**, 063602 (2008) [arXiv:0805.3226v4 [cond-mat.stat-mech]].
- [22] C. Cao, E. Elliott, H. Wu and J.E. Thomas, *NJP* **13**, 075007 (2011).
- [23] C. Cercignani, *Rarefied gas dynamics*, Cambridge University Press, Cambridge, 2000
- [24] Y. Sone, *Kinetic theory and fluid dynamics*, Birkhauser, Boston, 2002
- [25] H. Struchtrup, *Macroscopic transport equations for rarefied gas flows*, Springer, New York, 2005
- [26] M. Braby, J. Chao and T. Schafer, “Thermal Conductivity and Sound Attenuation in Dilute Atomic Fermi Gases,” *Phys. Rev. A* **82**, 033619 (2010) [arXiv:1003.2601 [cond-mat.quant-gas]].
- [27] M. A. Escobedo, M. Mannarelli and C. Manuel, “Bulk viscosities for cold Fermi superfluids close to the unitary limit,” *Phys. Rev. A* **79**, 063623 (2009) [arXiv:0904.3023 [cond-mat.quant-gas]].
- [28] J. R. Engelbrecht, M. Randeria and C. A. R. Sa de Melo, *Phys. Rev. B* **55**, 15153 (1997).
- [29] E. Gubankova, M. Mannarelli and R. Sharma, *Annals Phys.* **325**, 1987 (2010) [arXiv:0804.0782 [cond-mat.supr-con]]; R. Anglani, M. Mannarelli and M. Ruggieri, *New J. Phys.* **13**, 055002 (2011) [arXiv:1101.4277 [hep-ph]].
- [30] R. B. Diener, R. Sensarma and M. Randeria, *Phys. Rev. A* **77**, 023626 (2008) .
- [31] L. Salasnich and F. Toigo, *Phys. Rev. A* **78**, 053626 (2008); *Phys. Rev. A* **82**, 059902(E) (2010)
- [32] R. Combescot, M. Y. Kagan and S. Stringari, *Phys. Rev. A* **74**, 042717 (2006) [cond-mat/0607493 [cond-mat.other]].
- [33] M. J. H. Ku, A. T. Sommer, L. W. Cheuk and M. W. Zwierlein, *Science* **335**, 563 (2012).
- [34] R. Haussmann, W. Rantner, S. Cerrito and W. Zwerger, *Phys. Rev. A* **75**, 023610 (2007) [cond-mat/0608282 [cond-mat.stat-mech]].
- [35] P. B. Arnold, J. E. Drut and D. T. Son, *Phys. Rev. A* **75**, 043605 (2007) [cond-mat/0608477 [cond-mat.other]].
- [36] G. Bertsch, Proceedings of the 10th International Conference on Recent Progress in Many-Body Theories, R. F. Bishop et al., Eds. (World Scientific, Singapore, 2000)
- [37] M. A. Escobedo and C. Manuel, *Phys. Rev. A* **82**, 023614 (2010) [arXiv:1004.2567 [cond-mat.quant-gas]].
- [38] J. L. Mañes and M. A. Valle, *Annals of Physics* **324**, 1136 (2009) [arXiv:0810.3797 [cond-mat.other]].
- [39] R. Haussmann, M. Punk and W. Zwerger, *Phys. Rev. A* **80**, 063612 (2009) [arXiv:0904.1333v3 [cond-mat.quant-gas]]
- [40] C. Manuel and L. Tolos, *Phys. Rev. D* **84**, 123007 (2011) [arXiv:1110.0669 [astro-ph.SR]].
- [41] A. A. Zadorozhko, É. Y. Rudavskii, V. K. Chagovets, G. A. Sheshin, and Y. A. Kitsenko, *Low Temperature Physics*, **35**, 100 (2009).
- [42] G. Rupak and T. Schafer, *Phys. Rev. A* **76**, 053607 (2007) [arXiv:0707.1520 [cond-mat.other]].
- [43] J. Hojgaard et al, *Phys. Rev.* **185**, 323 (1969).
- [44] E. Burovski, N. Prokof'ev, B. Svistunov and M. Troyer, *Phys. Rev. Lett.* **96**, 160402 (2006) [cond-mat/0602224 [cond-mat.stat-mech]].
- [45] E. Burovski, E. Kozik, N. Prokof'ev, B. Svistunov and M. Troyer, *Phys. Rev. Lett.* **101**, 090402 (2008) [0805.3047v2 [cond-mat.str-el]].
- [46] M. Bartenstein, A. Altmeyer, S. Riedl, R. Geursen, S. Jochim, C. Chin, J. Hecker Denschlag, R. Grimm, A. Simoni, E. Tiesinga, C. J. Williams and P. S. Julienne, *Phys. Rev. Lett.* **94**, 103201 (2005).
- [47] C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schafer and J. E. Thomas, *Science* **331**, 58 (2011) [arXiv:1007.2625 [cond-mat.quant-gas]].
- [48] G. M. Bruun and H. Smith, *Phys. Rev. A* **75**, 043612 (2007).
- [49] G. Wlazlowski, P. Magierski and J. E. Drut, *Phys. Rev. Lett.* **109**, 020406 (2012)
- [50] K. Dusling and T. Schaefer, [arXiv:1207.5068[cond-mat.quant-gas]].
- [51] C. J. Pethick, H. Smith and P. Bhattacharyya, *Phys. Rev. Lett.* **34**, 643 (1975).
- [52] H. Guo, D. Wulin, C. -C. Chien and K. Levin, *Phys. Rev. Lett.* **107**, 020403 (2011) [arXiv:1008.0423 [cond-mat.quant-gas]]; *ibid*, *New. J. Phys.* **13**, 075011 (2011)) [arXiv:1009.4678 [cond-mat.supr-con]]
- [53] J. Hojgaard et al, *J. Low Temperature Physics*, **41**, 473 (1980).
- [54] B. N. Esel'Son, O. S. Nosovitskaya, L. A. Pogorelov, and V. I. Sobolev, *ZhETF Pis ma Redaktsiiu*, **31**, 34 (1980).
- [55] M. Niemetz and W. Schoepe, *J. Low Temperature Physics*, **135**, 447 (2004).
- [56] N.W. Ashcroft, D. Mermin, *Solid State Physics*, Harcourt College Publisher, USA, 1976.